

THE STRUCTURES ON THE UNIVERSAL ENVELOPING ALGEBRAS OF DIFFERENTIAL GRADED POISSON HOPF ALGEBRAS

MENTTIAN GUO, XIANGUO HU, JIAFENG LÜ*, AND XINGTING WANG

ABSTRACT. In this paper, the so-called differential graded (DG for short) Poisson Hopf algebra is introduced, which can be considered as a natural extension of Poisson Hopf algebras in the differential graded setting. The structures on the universal enveloping algebras of differential graded Poisson Hopf algebras are discussed.

1. INTRODUCTION

Poisson algebras appear naturally in Hamiltonian mechanics, and play a central role in the study of Poisson geometry and quantum groups. With the development of Poisson algebras in the past decades, many important generalizations have been obtained in both commutative and noncommutative settings: Poisson PI algebras [10], graded Poisson algebras [3], double Poisson algebras [20], Quiver Poisson algebras [22], noncommutative Leibniz-Poisson algebras [1], Left-right noncommutative Poisson algebras [2] and differential graded Poisson algebras [8], etc. One of most interesting features in this area is the Poisson universal enveloping algebra, which was first introduced by Oh [12] in order to describe the category of Poisson modules. Since then, Poisson universal enveloping algebras have been studied in a series of papers [15, 19, 21]. In particular, the third author and the fourth author of the present paper studied the universal enveloping algebras of Poisson Ore-extensions and DG Poisson algebra [7, 8].

We know that, the notion of Poisson Hopf algebras originally arises from Poisson geometry and quantum groups. For instance, the coordinate ring of a Poisson algebraic group is a Poisson Hopf algebra [5]. Recently, Poisson Hopf algebras are studied by many authors from different perspectives [6, 13, 14, 17]. In [13], Oh developed the theory of universal enveloping algebras for Poisson algebras, and then proved that the universal enveloping algebra A^e of a Poisson Hopf algebra A is a Hopf algebra. In addition, the last two authors of this paper studied the most basic properties for universal enveloping algebras of Poisson Hopf algebras [6], which would help us to understand Hopf algebras in general.

Motivated by the notion of differential graded Poisson algebra, our aim in this paper is to study Poisson Hopf algebras and their universal enveloping algebras in the DG setting. Roughly speaking, a DG Poisson Hopf algebra is a DG Poisson algebra together with a Hopf structure satisfying certain compatible conditions; see Definition 3.1. As for universal enveloping algebras, there are many equivalent ways to define the universal enveloping algebra of a DG Poisson algebra, among which we choose to use the universal property; see Definition 2.3.

Note that the universal enveloping algebra A^e of a Poisson bialgebra A is a bialgebra, as an extension, we prove the following result:

Theorem 1.1. *If $(A, u, \eta, \Delta, \varepsilon, \{\cdot, \cdot\}, d)$ is a DG Poisson bialgebra, then*

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*corresponding author.

$$(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e, d^e)$$

is a DG bialgebra such that

$$\begin{aligned} \Delta^e m &= (m \otimes m) \Delta & \Delta^e h &= (m \otimes h + h \otimes m) \Delta \\ \varepsilon^e m &= \varepsilon & \varepsilon^e h &= 0. \end{aligned}$$

It should be noted that the universal enveloping algebra A^e of a DG Poisson Hopf algebra A is just a DG bialgebra. But in some special cases, A^e can be endowed with the Hopf structure, such that A^e becomes a DG Hopf algebra:

Theorem 1.2. *Let $(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ be a DG Poisson Hopf algebra. Suppose that A^{eop} is the DG opposite algebra of A^e . Then:*

(1) *There exist a DG algebra homomorphism $S^e : A^e \rightarrow A^{eop}$ such that*

$$S^e m = mS, \quad S^e h = hS.$$

(2) *$\{S(a_{(1)}), a_{(2)}\} = 0$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in A$, if and only if*

$$(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e, S^e, d^e)$$

is a DG Hopf algebra.

The paper is organized as follows. In Section 2, we briefly review some basic concepts and results related to DG algebras, DG Poisson algebras, DG Hopf algebras and universal enveloping algebras of DG Poisson algebras. Section 3 is devoted to the study of some definitions and properties of DG Poisson Hopf algebras and universal enveloping algebras. In particular, we show that there is a natural DG bialgebra structure on the universal enveloping algebra A^e for a DG Poisson bialgebra A .

Throughout the whole paper, \mathbb{Z} denotes the set of integers, \mathbb{k} denotes a base field of characteristic zero unless otherwise stated, all (graded) algebras are assumed to have an identity and all (graded) modules are assumed to be unitary. We always take the grading to be \mathbb{Z} -graded.

2. PRELIMINARIES

In this section, we will recall some definitions and results of DG Poisson algebras, universal enveloping algebras and DG Hopf algebras.

2.1. DG Poisson algebras. By a graded algebra A we mean a \mathbb{Z} -graded algebra (A, u, η) , where $u : A \otimes A \rightarrow A$ and $\eta : \mathbb{k} \rightarrow A$ are called the multiplication and unit of A , respectively. For convenience, we shall write $u(a \otimes b)$ as ab , $\forall a, b \in A$, whenever this does not cause confusion. A DG algebra is a graded algebra with a \mathbb{k} -linear homogeneous map $d : A \rightarrow A$ of degree 1, which is also a graded derivation. Let A, B be two DG algebras and $f : A \rightarrow B$ be a graded algebra map of degree zero. Then f is called a DG algebra map if f commutes with the differentials.

Definition 2.1. Let (A, \cdot) be a graded \mathbb{k} -algebra. If there is a \mathbb{k} -linear map

$$\{\cdot, \cdot\} : A \otimes A \rightarrow A$$

of degree p such that

- (i) $(A, \{\cdot, \cdot\})$ is a graded Lie algebra. That is to say, we have
 - (ia) $\{a, b\} = -(-1)^{(|a|+p)(|b|+p)}\{b, a\}$;
 - (ib) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+p)(|b|+p)}\{b, \{a, c\}\}$,
- (ii) (graded commutativity): $a \cdot b = (-1)^{|a||b|}b \cdot a$;
- (iii) (biderivation property): $\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(|a|+p)|b|}b \cdot \{a, c\}$,

for any homogeneous elements $a, b, c \in A$, then A is called a graded Poisson algebra [3]. If in addition, there is a \mathbb{k} -linear homogeneous map $d : A \rightarrow A$ of degree 1 such that $d^2 = 0$ and

$$(iv) \quad d(\{a, b\}) = \{d(a), b\} + (-1)^{(|a|+p)}\{a, d(b)\};$$

$$(v) \quad d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b),$$

for any homogeneous elements $a, b \in A$, then A is called a DG Poisson algebra, which is usually denoted by $(A, \cdot, \{\cdot, \cdot\}, d)$, or simply by $(A, \{\cdot, \cdot\}, d)$ or A if no confusions arise.

Lemma 2.2. [8] *Let $(A, \cdot, \{\cdot, \cdot\}_A, d_A)$ and $(B, *, \{\cdot, \cdot\}_B, d_B)$ be any two DG Poisson algebras with Poisson brackets of degree p . Then $(A \otimes B, \star, \{\cdot, \cdot\}, d)$ is a DG Poisson algebra, where*

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &:= (-1)^{|a'| |b|} (a \cdot a') \otimes (b * b'), \\ d(a \otimes b) &:= d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b), \\ \{a \otimes b, a' \otimes b'\} &:= (-1)^{(|a'|+p)|b|} \{a, a'\}_A \otimes (b * b') + (-1)^{(|b|+p)|a'|} (a \cdot a') \otimes \{b, b'\}_B, \end{aligned}$$

for any homogeneous elements $a, a' \in A$ and $b, b' \in B$.

For DG Poisson algebras A and B , a graded algebra homomorphism $\phi : A \rightarrow B$ is said to be a DG Poisson algebra homomorphism if $\phi \circ d_A = d_B \circ \phi$ and $\phi(\{a, b\}_A) = \{\phi(a), \phi(b)\}_B$ for all homogeneous elements $a, b \in A$. We denote by **DG(P)A** the category of DG (Poisson) algebras whose morphism space consists of DG (Poisson) algebras homomorphism.

For a DG algebra B , assume throughout the paper that, B_P will be the DG Poisson algebra B with the standard graded Lie bracket $[a, b] = ab - (-1)^{(|a|+p)(|b|+p)} ba$ for all homogeneous elements $a, b \in B$, where p is the degree of the Poisson bracket for B .

Let us review the definition of a universal enveloping algebra of $A \in \mathbf{DGPA}$.

Definition 2.3. For a DG Poisson algebra A , a quadruple $(A^e, \alpha, \beta, \partial)$, where $A^e \in \mathbf{DGA}$, $\alpha : (A, d) \rightarrow (A^e, \partial)$ is a DG algebra map and $\beta : (A, \{\cdot, \cdot\}, d) \rightarrow (A_P^e, [\cdot, \cdot], \partial)$ is a DG Lie algebra map such that

$$\begin{aligned} \alpha(\{a, b\}) &= \beta(a)\alpha(b) - (-1)^{(|a|+p)|b|} \alpha(b)\beta(a), \\ \beta(ab) &= \alpha(a)\beta(b) + (-1)^{|a||b|} \alpha(b)\beta(a), \end{aligned}$$

for any homogeneous elements $a, b \in A$ and p is the degree of the Poisson bracket for A , is called the universal enveloping algebra of A if $(A^e, \alpha, \beta, \partial)$ satisfies the following: if (D, δ) is a DG algebra, f is a DG algebra map from (A, d) into (D, δ) and g is a DG Lie algebra map from $(A, \{\cdot, \cdot\}, d)$ into $(D_P, [\cdot, \cdot], \delta)$ such that

$$\begin{aligned} f(\{a, b\}) &= g(a)f(b) - (-1)^{(|a|+p)|b|} f(b)g(a), \\ g(ab) &= f(a)g(b) + (-1)^{|a||b|} f(b)g(a), \end{aligned}$$

for all $a, b \in A$, then there exists a unique DG algebra map $\phi : (A^e, \partial) \rightarrow (D, \delta)$, such that $\phi\alpha = f$ and $\phi\beta = g$.

For every DG Poisson algebra A , there exists a unique universal enveloping algebra A^e up to isomorphic, that A^e is generated by $m(A)$ and $h(A)$ by [8] and that $h(1) = 0$. Note that A^e is a DG algebra, such that a \mathbb{k} -vector space M is a DG Poisson A -module if and only if M is a DG A^e -module.

As an application of the “universal property” of universal enveloping algebras, we have

Lemma 2.4. [8] *For any $A, B \in \mathbf{DGPA}$, suppose that (A^e, m_A, h_A) and (B^e, m_B, h_B) are the universal enveloping algebra of A and B , respectively. Then $(A^e \otimes B^e, m_A \otimes m_B, m_A \otimes h_B + (-1)^{p|B|} h_A \otimes m_B)$ is the universal enveloping algebra for $A \otimes B$, where p is the degree of the Poisson bracket for both A and B . We use $(-1)^{p|B|} h_A \otimes m_B$ to mean that*

$$((-1)^{p|B|} h_A \otimes m_B)(a \otimes b) = (-1)^{p|b|} h_A(a) \otimes m_B(b)$$

for any $a \in A, b \in B$.

Lemma 2.5. *Let (A^e, m_A, h_A) and (B^e, m_B, h_B) be universal enveloping algebras for DG Poisson algebras A and B respectively. Assume that p is the degree of the Poisson bracket for both A and B . If*

$\phi : A \rightarrow B$ is a DG Poisson homomorphism, then there exists a unique DG algebra homomorphism $\phi^e : A^e \rightarrow B^e$ such that $\phi^e m_A = m_B \phi$ and $\phi^e h_A = h_B \phi$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ m_A, h_A \downarrow & & \downarrow m_B, h_B \\ A^e & \xrightarrow{\phi^e} & B^e \end{array}$$

Proof. Note that m_B is a DG algebra map and h_B is a DG Lie algebra map such that

$$m_B(\{b, b'\}) = h_B(b)m_B(b') - (-1)^{(|b|+p)|b'|} m_B(b')h_B(b),$$

$$h_B(bb') = m_B(b)h_B(b') + (-1)^{|b||b'|} m_B(b')h_B(b).$$

We have

$$m_B\phi(aa') = m_B(\phi(a)\phi(a')) = m_B\phi(a)m_B\phi(a'),$$

$$h_B\phi(\{a, a'\}) = h_B(\{\phi(a), \phi(a')\}) = [h_B\phi(a), h_B\phi(a')],$$

$$m_B\phi(\{a, a'\}) = m_B(\{\phi(a), \phi(a')\}) = h_B\phi(a)m_B\phi(a') - (-1)^{(|a|+p)|a'|} m_B\phi(a')h_B\phi(a)$$

and

$$h_B\phi(aa') = h_B(\phi(a)\phi(a')) = m_B\phi(a)h_B\phi(a') + (-1)^{|a||a'|} m_B\phi(a')h_B\phi(a),$$

since $\phi : A \rightarrow B$ is a DG Poisson homomorphism. Further, it is easy to see

$$m_B\phi(d_A(a)) = m_B d_B(\phi(a)) = d_{B^e} m_B\phi(a)$$

and

$$h_B\phi(d_A(a)) = h_B d_B(\phi(a)) = d_{B^e} h_B\phi(a).$$

Thus there exists a unique DG algebra homomorphism ϕ^e from A^e into B^e such that $\phi^e m_A = m_B \phi$ and $\phi^e h_A = h_B \phi$. \square

2.2. DG Hopf algebras. In this subsection, we recall some definitions and properties of graded Hopf algebras and DG Hopf algebras.

In the remainder of the paper, the twisting map $T : V \otimes W \rightarrow W \otimes V$ will frequently appear. It is defined for homogeneous elements $v \in V$ and $w \in W$ by

$$T(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

and extends to all elements of V and W through linearity.

A graded coalgebra C over \mathbb{k} is a \mathbb{Z} -graded vector space with the graded vector space homomorphisms of degree 0 $\Delta : C \rightarrow C \otimes C$ and $\varepsilon : C \rightarrow \mathbb{k}$ such that the following diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow I \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes I} & C \otimes C \otimes C \end{array} \quad \begin{array}{ccc} & C \otimes C & \\ I \otimes \varepsilon \swarrow & \uparrow \Delta & \searrow \varepsilon \otimes I \\ C \otimes \mathbb{k} & & \mathbb{k} \otimes C \\ \cong \swarrow & \downarrow C & \searrow \cong \\ & C & \end{array}$$

are commutative, where $I : C \rightarrow C$ is the identity homomorphism, Δ and ε are called the comultiplication and counit of C , respectively. Note that the commutativity of the first diagram is equivalent to

$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$, called the coassociativity of Δ . Further, C is graded cocommutative if $\Delta = T\Delta$, where $T : C \otimes C \rightarrow C \otimes C$ is the twisting morphism.

For any $c \in C$, we write $\Delta(c) = c_{(1)} \otimes c_{(2)}$, which is basically the Sweedler's notation with the summation sign Σ omitted. In this notation, the comultiplication and counit property may be expressed as

$$\begin{aligned} (\Delta \otimes I)\Delta(c) &= (I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \\ c &= \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)}), \end{aligned}$$

for any homogeneous elements $c \in C$, respectively.

A homomorphism of graded coalgebras $f : A \rightarrow B$ is a homomorphism of graded vector spaces such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \otimes A \\ f \downarrow & & \downarrow f \otimes f \\ B & \xrightarrow{\Delta_B} & B \otimes B \end{array} \quad \begin{array}{ccc} & & A \\ & \swarrow \varepsilon_A & \downarrow f \\ \mathbb{k} & & B \\ & \nwarrow \varepsilon_B & \end{array}$$

are commutative.

Let H be a graded algebra with multiplication u and unit η , and at the same time a graded coalgebra with comultiplication Δ and counit ε . Then H is called a graded bialgebra provided that one of the following equivalent conditions are satisfied:

- (i) Δ and ε are graded algebra homomorphisms;
- (ii) u and η are graded coalgebra homomorphisms.

Further, if H admits a graded vector space homomorphism $S : H \rightarrow H$ of degree 0, which satisfies the following defining relation

$$u(I \otimes S)\Delta = u(S \otimes I)\Delta = \eta\varepsilon,$$

then H is called a graded Hopf algebra, and S is called the antipode of H .

The antipode S has the following properties [4, 9, 16].

Lemma 2.6. *Let H be a graded Hopf algebra and S its antipode; then*

- (1) $S \cdot u = u \cdot T \cdot (S \otimes S)$,
- (2) $S \cdot \eta = \eta$,
- (3) $\varepsilon \cdot S = \varepsilon$,
- (4) $T \cdot (S \otimes S) \cdot \Delta = \Delta \cdot S$,
- (5) *if H is graded commutative or graded cocommutative, then $S \cdot S = I$,*

where $I : H \rightarrow H$ is the identity morphism and $T : H \otimes H \rightarrow H \otimes H$ is the twisting morphism.

Definition 2.7. [18] Let H be a graded \mathbb{k} -vector space. If there is a \mathbb{k} -linear homogeneous map $d : H \rightarrow H$ of degree 1 such that $d^2 = 0$ and

- (i) (H, u, η, d) is a DG algebra. That is to say, we have
 - (ia) (H, u, η) is a graded algebra;
 - (ib) d is a (algebra) derivation of degree 1, that means,
$$d(ab) = d(a)b + (-1)^{|a|}ad(b),$$
for any homogeneous elements $a, b \in H$.
- (ii) $(H, \Delta, \varepsilon, d)$ is a DG coalgebra. That is, we have
 - (iia) (H, Δ, ε) is a graded coalgebra;

- (iib) d is a coderivation of degree 1, that means, $\varepsilon d = 0$ and

$$\Delta d = (d \otimes I + T(d \otimes I)T)\Delta.$$

- (iii) $(H, u, \eta, \Delta, \varepsilon)$ is a graded bialgebra.

Then H is called a DG bialgebra. Moreover, if $(H, u, \eta, \Delta, \varepsilon, S)$ is a graded Hopf algebra, then H is called a DG Hopf algebra, which is usually denoted by $(H, u, \eta, \Delta, \varepsilon, S, d)$.

3. DG POISSON HOPF ALGEBRAS AND UNIVERSAL ENVELOPING ALGEBRAS

Now we will introduce the notion of DG Poisson Hopf algebras and prove that the universal enveloping algebra A^e of a DG Poisson bialgebra A is a DG bialgebra.

Definition 3.1. Let A be a graded \mathbb{k} -vector space. If there is a \mathbb{k} -linear map

$$\{\cdot, \cdot\} : A \otimes A \rightarrow A$$

of degree p such that:

- (i) $(A, u, \eta, \{\cdot, \cdot\})$ is a graded Poisson algebra;
- (ii) $(A, u, \eta, \Delta, \varepsilon)$ is a graded bialgebra;
- (iii) $\Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A \otimes A}$ for all $a, b \in A$, where the Poisson bracket $\{\cdot, \cdot\}_{A \otimes A}$ on $A \otimes A$ is defined by

$$\{a \otimes a', b \otimes b'\}_{A \otimes A} = (-1)^{(|b|+p)|a'|}\{a, b\} \otimes a'b' + (-1)^{(|a'|+p)|b|}ab \otimes \{a', b'\}$$

for any homogeneous elements $a, b, a', b' \in A$.

Then A is called a graded Poisson bialgebra. If in addition, there is a \mathbb{k} -linear homogeneous map $d : A \rightarrow A$ of degree 1 such that $d^2 = 0$ and

- (iva) $d(\{a, b\}) = \{d(a), b\} + (-1)^{(|a|+p)}\{a, d(b)\}$;
- (ivb) $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b)$;
- (ivc) $\varepsilon d = 0$ and $\Delta d(a) = d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|}a_{(1)} \otimes d(a_{(2)})$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$,

for any homogeneous elements $a, b \in A$, then $(A, u, \eta, \Delta, \varepsilon, \{\cdot, \cdot\}, d)$ is called a DG Poisson bialgebra. Further, if A admits a graded vector space homomorphism $S : A \rightarrow A$ of degree 0 which satisfies the following defining relation

$$u(I \otimes S)\Delta = u(S \otimes I)\Delta = \eta\varepsilon.$$

Then A is called a DG Poisson Hopf algebra, which is usually denoted by

$$(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d).$$

From now on, we would like to use the degree of the Poisson bracket is zero in our definition of DG Poisson Hopf algebra, but the result obtained in this paper are also true for DG Poisson Hopf algebra of degree p with some expected signs, where $p \in \mathbb{Z}$ is the degree of the Poisson bracket.

As a generalization of Lemma 2.2, we have

Proposition 3.2. Let $(A, u_A, \eta_A, \Delta_A, \varepsilon_A, S_A, \{\cdot, \cdot\}_A, d_A)$ and $(B, u_B, \eta_B, \Delta_B, \varepsilon_B, S_B, \{\cdot, \cdot\}_B, d_B)$ be any two DG Poisson Hopf algebras with Poisson brackets of degree 0. Then

$$(A \otimes B, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$$

is a DG Poisson Hopf algebra, where

$$\begin{aligned} u((a \otimes b) \otimes (a' \otimes b')) &:= (-1)^{|a'| |b|} aa' \otimes bb', & \eta(1_{\mathbb{k}}) &:= 1_A \otimes 1_B, \\ \Delta(a \otimes b) &:= (-1)^{|a_{(2)}| |b_{(1)}|} a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}, & \varepsilon(a \otimes b) &:= \varepsilon_A(a) \varepsilon_B(b), \\ d(a \otimes b) &:= d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b), & S(a \otimes b) &:= S_A(a) \otimes S_B(b), \\ \{a \otimes b, a' \otimes b'\} &:= (-1)^{|a'| |b|} (\{a, a'\}_A \otimes bb' + aa' \otimes \{b, b'\}_B), \end{aligned}$$

for any homogeneous elements $a, a' \in A$ and $b, b' \in B$. As usual, we write $\Delta(a) = a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = b_{(1)} \otimes b_{(2)}$.

Proof. It is verified routinely that $(A \otimes B, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ is a DG Poisson Hopf algebra. \square

Now we give some examples of DG Poisson Hopf algebras.

Example 3.3. Let $(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ be any DG Poisson Hopf algebra with Poisson brackets of degree 0. Then

$$(A^{op}, u^{op}, \eta, \Delta^{op}, \varepsilon, S, \{\cdot, \cdot\}^{op}, d)$$

is also a DG Poisson Hopf algebra, where

$$u^{op}(a \otimes b) = (-1)^{|a||b|} b \cdot a = a \cdot b = u(a \otimes b),$$

$$\Delta^{op} = T\Delta,$$

$$\{a, b\}^{op} = (-1)^{|a||b|} \{b, a\} = -\{a, b\},$$

for any homogeneous elements $a, b \in A$, and $T : A \otimes A \rightarrow A \otimes A$ is the twisting morphism.

From the constructions of opposite algebra and tensor product of DG Poisson Hopf algebras, we have the following observation:

Proposition 3.4. Let A and B be any two DG Poisson Hopf algebras. Then

$$(A \otimes B)^{op} = A^{op} \otimes B^{op}.$$

In [6], there are a lot of examples about Poisson Hopf algebras, which can be considered as DG Poisson Hopf algebras concentrated in degree 0 with trivial differential. For instance, we have the following example.

Example 3.5. Assume that the base field \mathbb{k} has characteristic $p > 2$. Let

$$B = \mathbb{k}[x, y, z]/(x^p, y^p, z^p)$$

be the restricted symmetric algebra with three variables. Then B becomes a Poisson Hopf algebra via

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

$$\Delta(y) = y \otimes 1 + 1 \otimes y,$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z - 2x \otimes y,$$

$$S(x) = -x, \quad S(y) = -y, \quad S(z) = -z - 2xy,$$

$$\{x, y\} = y, \quad \{y, z\} = y^2, \quad \{x, z\} = z.$$

In the above example, one should view it as a Poisson version of the Hopf algebra in [11]. Similarly, we can obtain other Poisson Hopf algebras from connected Hopf algebras in [11].

Example 3.6. Let (L, d_L) be a finite dimensional DG Lie algebra over \mathbb{k} with standard graded Lie bracket $[\cdot, \cdot]$ and let $(S(L), u, \eta)$ be the graded symmetric algebra of L , i.e.,

$$S(L) := \frac{T(L)}{(a \otimes b - (-1)^{|a||b|} b \otimes a)},$$

for any $a, b \in L$, where u is a multiplication and η is a unit. The differential d_L of L can be extended to the graded symmetric algebra $S(L)$ such that $S(L)$ becomes a graded commutative DG algebra and denote by d the differential in $S(L)$. Here, we consider the total grading on $S(L)$ coming from the grading of L . Moreover, the Lie bracket on L also can be extended to a Poisson bracket on $S(L)$ by graded Jacobi identity such that

$$\{a, b\}_{S(L)} := [a, b],$$

for any $a, b \in L$. Hence, the graded symmetric algebra $S(L)$ over L has a natural DG Poisson algebra structure.

Now, we introduce the graded Hopf structure for $S(L)$. Denote by Δ the familiar comultiplication $\Delta : S(L) \rightarrow S(L) \otimes S(L)$ such that

$$\begin{aligned}\Delta(a) &= a \otimes 1 + 1 \otimes a, \quad \forall a \in L, \\ \Delta(uv) &= \Delta(u)\Delta(v), \quad \forall u, v \in S(L)\end{aligned}$$

and let the morphism $\varepsilon : S(L) \rightarrow \mathbb{k}$ be defined by

$$\begin{aligned}\varepsilon(1) &= 1, \\ \varepsilon(a) &= 0, \quad \forall a \in L, \\ \varepsilon(uv) &= \varepsilon(u)\varepsilon(v), \quad \forall u, v \in S(L),\end{aligned}$$

then $S(L)$ constitutes a \mathbb{Z} -graded bialgebra with comultiplication Δ and counit ε . Fix a \mathbb{k} -basis x_1, x_2, \dots, x_n of L . Note that $S(L)$ is the graded commutative polynomial ring $k[x_1, x_2, \dots, x_n]$, we can prove the formula $\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}$ by using the induction on degree of homogeneous elements of $S(L)$. In fact, we have

$$\begin{aligned}\Delta(\{x_i, x_j\}_{S(L)}) &= \Delta(x_i x_j - (-1)^{|x_i||x_j|} x_j x_i) \\ &= \Delta(x_i)\Delta(x_j) - (-1)^{|x_i||x_j|} \Delta(x_j)\Delta(x_i) \\ &= (x_i \otimes 1 + 1 \otimes x_i)(x_j \otimes 1 + 1 \otimes x_j) - (-1)^{|x_i||x_j|} (x_j \otimes 1 + 1 \otimes x_j)(x_i \otimes 1 + 1 \otimes x_i) \\ &= x_i x_j \otimes 1 + x_i \otimes x_j + (-1)^{|x_i||x_j|} (x_j \otimes x_i) + 1 \otimes x_i x_j \\ &\quad - (-1)^{|x_i||x_j|} (x_j x_i \otimes 1 + x_j \otimes x_i + (-1)^{|x_i||x_j|} (x_i \otimes x_j) + 1 \otimes x_j x_i) \\ &= x_i x_j \otimes 1 + 1 \otimes x_i x_j - (-1)^{|x_i||x_j|} (x_j x_i \otimes 1) - (-1)^{|x_i||x_j|} (1 \otimes x_j x_i)\end{aligned}$$

and

$$\begin{aligned}\{\Delta(x_i), \Delta(x_j)\} &= \{x_i \otimes 1 + 1 \otimes x_i, x_j \otimes 1 + 1 \otimes x_j\} \\ &= \{x_i, x_j\} \otimes 1 + 1 \otimes \{x_i, x_j\} \\ &= x_i x_j \otimes 1 - (-1)^{|x_i||x_j|} (x_j x_i \otimes 1) + 1 \otimes x_i x_j - (-1)^{|x_i||x_j|} (1 \otimes x_j x_i),\end{aligned}$$

and so we have $\Delta(\{x_i, x_j\}) = \{\Delta(x_i), \Delta(x_j)\}$. For any monomials $a, b, c \in S(L)$, assume that $\Delta(\{a, c\}) = \{\Delta(a), \Delta(c)\}$ and $\Delta(\{b, c\}) = \{\Delta(b), \Delta(c)\}$. Now we have

$$\begin{aligned}\Delta(\{ab, c\}) &= \Delta(a\{b, c\} + (-1)^{|a||b|} b\{a, c\}) \\ &= \Delta(a)\Delta(\{b, c\}) + (-1)^{|a||b|} \Delta(b)\Delta(\{a, c\}) \\ &= \Delta(a)\{\Delta(b), \Delta(c)\} + (-1)^{|a||b|} \Delta(b)\{\Delta(a), \Delta(c)\} \\ &= \{\Delta(ab), \Delta(c)\}.\end{aligned}$$

Applying the induction and biderivation property, we have

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\},$$

for all $a, b \in S(L)$. In fact, it is also verified easily using the induction on degree of homogeneous elements of $S(L)$ that $\Delta d(a) = d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|} a_{(1)} \otimes d(a_{(2)})$, $\varepsilon d(a) = 0$ and $\varepsilon(\{a, b\}) = 0$ for all $a, b \in S(L)$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Hence $(S(L), u, \eta, \Delta, \varepsilon, \{\cdot, \cdot\}, d)$ is a DG Poisson bialgebra. In order to finish the proof, it suffice to prove that

$$u(S \otimes I)\Delta = u(I \otimes S)\Delta = \eta\varepsilon,$$

where the \mathbb{k} -linear homogeneous map $S : S(L) \rightarrow S(L)$ of degree 0 is defined by

$$\begin{aligned} S(1) &= 1, \\ S(a) &= -a, \quad \forall a \in L, \\ S(uv) &= (-1)^{|u||v|} S(v)S(u) \end{aligned}$$

for homogeneous elements of $S(L)$ which extends to all $S(L)$ by linearity. Then we have

$$u(S \otimes I)\Delta(x_i) = u(S \otimes I)(x_i \otimes 1 + 1 \otimes x_i) = S(x_i) + x_i = -x_i + x_i = 0 = \eta\varepsilon(x_i)$$

and

$$u(I \otimes S)\Delta(x_i) = u(I \otimes S)(x_i \otimes 1 + 1 \otimes x_i) = x_i + S(x_i) = x_i - x_i = 0 = \eta\varepsilon(x_i).$$

Now it suffices to prove the formula is true for ab provided that it is true for any homogeneous elements $a, b \in S(L)$. Thus, assume that we have the following two equations:

$$u(S \otimes I)\Delta(a) = \eta\varepsilon(a) \text{ and } u(S \otimes I)\Delta(b) = \eta\varepsilon(b),$$

for any homogeneous elements $a, b \in S(L)$. Then we have

$$\begin{aligned} &u(S \otimes I)\Delta(ab) \\ &= u(S \otimes I)(\Delta(a)\Delta(b)) = u(S \otimes I)((-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}) \\ &= (-1)^{|a_{(2)}||b_{(1)}|} S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = (-1)^{|a_{(2)}||b_{(1)}|+|a_{(1)}||b_{(1)}|} S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)} \\ &= S(a_{(1)})S(a_{(2)})S(b_{(1)})b_{(2)} = \eta\varepsilon(a)\eta\varepsilon(b) = \eta\varepsilon(ab). \end{aligned}$$

Similarly, we can prove the formula $u(I \otimes S)\Delta = \eta\varepsilon$ is true, which complete the proof.

Lemma 3.7. *Retain the notations of Example 3.6, we have that*

$$(3.1) \quad \{S(a_{(1)}), a_{(2)}\} = 0,$$

where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in S(L)$.

Proof. We proceed the proof using induction on the length of $a \in S(L)$. Observe that formula (1) is true on any $a \in L$ since

$$\{S(a_{(1)}), a_{(2)}\} = \{S(a), 1\} + \{S(1), a\} = \{-a, 1\} + \{1, a\} = 0,$$

where $a \in L$ and $\Delta(a) = a \otimes 1 + 1 \otimes a$.

In order to prove that formula (1) is true for all $a \in S(L)$. It suffices to prove the formula (1) is true for ab provided that it is true for any elements $a, b \in S(L)$. Thus, assume that we have the following two equations:

$$\{S(a_{(1)}), a_{(2)}\} = 0, \quad \{S(b_{(1)}), b_{(2)}\} = 0,$$

for any elements $a, b \in S(L)$. Note that

$$\Delta(ab) = \Delta(a)\Delta(b) = (a_{(1)} \otimes a_{(2)})(b_{(1)} \otimes b_{(2)}) = (-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}.$$

We have

$$\begin{aligned}
& \{S((ab)_{(1)}), (ab)_{(2)}\} \\
&= (-1)^{|a_{(2)}||b_{(1)}|} \{S(a_{(1)}b_{(1)}), a_{(2)}b_{(2)}\} \\
&= (-1)^{|a_{(2)}||b_{(1)}|} \{(-1)^{|a_{(1)}||b_{(1)}|} S(b_{(1)})S(a_{(1)}), a_{(2)}b_{(2)}\} \\
&= (-1)^{|a_{(1)}a_{(2)}||b_{(1)}|} [\{S(b_{(1)})S(a_{(1)}), a_{(2)}\}b_{(2)} + (-1)^{|a_{(1)}b_{(1)}||a_{(2)}|} a_{(2)}\{S(b_{(1)})S(a_{(1)}), b_{(2)}\}] \\
&= (-1)^{|a_{(1)}a_{(2)}||b_{(1)}|} [S(b_{(1)})\{S(a_{(1)}), a_{(2)}\}b_{(2)} + (-1)^{|a_{(1)}||a_{(2)}|} \{S(b_{(1)}), a_{(2)}\}S(a_{(1)})b_{(2)}] \\
&\quad + (-1)^{|a_{(1)}||a_{(2)}b_{(1)}|} [a_{(2)}S(b_{(1)})\{S(a_{(1)}), b_{(2)}\} + (-1)^{|a_{(1)}||b_{(2)}|} a_{(2)}\{S(b_{(1)}), b_{(2)}\}S(a_{(1)})] \\
&= (-1)^{|a_{(1)}||a_{(2)}b_{(1)}|} [(-1)^{|a_{(2)}||b_{(1)}|} \{S(b_{(1)}), a_{(2)}\}S(a_{(1)})b_{(2)} + a_{(2)}S(b_{(1)})\{S(a_{(1)}), b_{(2)}\}].
\end{aligned}$$

From the structure of DG Poisson Hopf algebra $S(L)$, we have

$$0 = \{\varepsilon(a)1_A, b\} = \{S(a_{(1)})a_{(2)}, b\} = S(a_{(1)})\{a_{(2)}, b\} + (-1)^{|b||a_{(2)}|} \{S(a_{(1)}), b\}a_{(2)}$$

and

$$0 = \{a, \varepsilon(b)1_A\} = \{a, S(b_{(1)})b_{(2)}\} = \{a, S(b_{(1)})\}b_{(2)} + (-1)^{|a||b_{(1)}|} S(b_{(1)})\{a, b_{(2)}\},$$

which imply that

$$S(a_{(1)})\{a_{(2)}, b\} = -(-1)^{|b||a_{(2)}|} \{S(a_{(1)}), b\}a_{(2)}$$

and

$$\{a, S(b_{(1)})\}b_{(2)} = -(-1)^{|a||b_{(1)}|} S(b_{(1)})\{a, b_{(2)}\}.$$

Thus

$$(-1)^{|a_{(1)}||a_{(2)}b_{(1)}|} [(-1)^{|a_{(2)}||b_{(1)}|} \{S(b_{(1)}), a_{(2)}\}S(a_{(1)})b_{(2)} + a_{(2)}S(b_{(1)})\{S(a_{(1)}), b_{(2)}\}] = 0$$

by using the above two formulas. Hence we have the conclusion. \square

Lemma 3.8. *If $(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ is a DG Poisson Hopf algebra, then $dS = Sd$, $S(\{a, b\}) = (-1)^{|a||b|} \{S(b), S(a)\}$ and the counit ε is a DG Poisson homomorphism.*

Proof. Firstly, since $\{a \otimes a', b \otimes b'\}_{A \otimes A} = (-1)^{|a'||b|} (\{a, b\} \otimes a' b' + a b \otimes \{a', b'\})$ and $\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}_{A \otimes A}$, for all $a, b, a', b' \in A$, we have

$$\begin{aligned}
\{a, b\} &= u(\varepsilon \otimes I)\Delta(\{a, b\}) \\
&= u(\varepsilon \otimes I)((-1)^{|b_{(1)}||a_{(2)}|} (\{a_{(1)}, b_{(1)}\} \otimes a_{(2)}b_{(2)} + a_{(1)}b_{(1)} \otimes \{a_{(2)}, b_{(2)}\})) \\
&= (-1)^{|b_{(1)}||a_{(2)}|} (\varepsilon(\{a_{(1)}, b_{(1)}\})a_{(2)}b_{(2)} + \varepsilon(a_{(1)}b_{(1)})\{a_{(2)}, b_{(2)}\}) \\
&= (-1)^{|b_{(1)}||a_{(2)}|} (\varepsilon(\{a_{(1)}, b_{(1)}\})a_{(2)}b_{(2)} + \{a, b\}),
\end{aligned}$$

for all $a, b \in A$, and so we have

$$\begin{aligned}
\varepsilon(\{a, b\}) &= (-1)^{|b_{(1)}||a_{(2)}|} \varepsilon(\varepsilon(\{a_{(1)}, b_{(1)}\})a_{(2)}b_{(2)} + \{a, b\}) \\
&= (-1)^{|b_{(1)}||a_{(2)}|} (\varepsilon(\{a_{(1)}, b_{(1)}\})\varepsilon(a_{(2)}b_{(2)}) + \varepsilon(\{a, b\})) \\
&= (-1)^{|b_{(1)}||a_{(2)}|} (\varepsilon(\{a, b\}) + \varepsilon(\{a, b\})),
\end{aligned}$$

hence $\varepsilon(\{a, b\}) = 0$. Note that $\varepsilon d = 0$, thus the counit ε is a DG Poisson homomorphism.

Secondly, since

$$0 = \{\varepsilon(a)1_A, b\} = \{S(a_{(1)})a_{(2)}, b\} = S(a_{(1)})\{a_{(2)}, b\} + (-1)^{|b||a_{(2)}|} \{S(a_{(1)}), b\}a_{(2)}$$

and

$$0 = \{a, \varepsilon(b)1_A\} = \{a, S(b_{(1)})b_{(2)}\} = \{a, S(b_{(1)})\}b_{(2)} + (-1)^{|a||b_{(1)}|} S(b_{(1)})\{a, b_{(2)}\},$$

we have

$$S(a_{(1)})\{a_{(2)}, b\} = -(-1)^{|b||a_{(2)}|}\{S(a_{(1)}), b\}a_{(2)}$$

and

$$\{a, S(b_{(1)})\}b_{(2)} = -(-1)^{|a||b_{(1)}|}S(b_{(1)})\{a, b_{(2)}\}.$$

Note that

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\} = (-1)^{|b_{(1)}||a_{(2)}|}(\{a_{(1)}, b_{(1)}\} \otimes a_{(2)}b_{(2)} + a_{(1)}b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}),$$

since A is a DG Poisson Hopf algebra. We have

$$0 = \varepsilon(\{a, b\})1_A = S(\{a, b\}_{(1)})\{a, b\}_{(2)} = (-1)^{|b_{(1)}||a_{(2)}|}(S(\{a_{(1)}, b_{(1)}\})a_{(2)}b_{(2)} + S(a_{(1)}b_{(1)})\{a_{(2)}, b_{(2)}\}).$$

Thus

$$\begin{aligned} & S(\{a_{(1)}, b_{(1)}\})a_{(2)}b_{(2)} \\ &= -S(a_{(1)}b_{(1)})\{a_{(2)}, b_{(2)}\} = -(-1)^{|a_{(1)}||b_{(1)}|}S(b_{(1)})S(a_{(1)})\{a_{(2)}, b_{(2)}\} \\ &= (-1)^{|a_{(1)}||b_{(1)}|+|b_{(2)}||a_{(2)}|}S(b_{(1)})\{S(a_{(1)}), b_{(2)}\}a_{(2)} = -(-1)^{|b_{(2)}||a_{(2)}|}\{S(a_{(1)}), S(b_{(1)})\}b_{(2)}a_{(2)} \\ &= -\{S(a_{(1)}), S(b_{(1)})\}a_{(2)}b_{(2)}. \end{aligned}$$

Therefore

$$\begin{aligned} & S(\{a, b\}) \\ &= S(\{a_{(1)}\varepsilon(a_{(2)}), b_{(1)}\varepsilon(b_{(2)})\}) = S(\{a_{(1)}, b_{(1)}\})\varepsilon(a_{(2)})\varepsilon(b_{(2)}) \\ &= S(\{a_{(1)}, b_{(1)}\})a_{(2)}S(a_{(3)})b_{(2)}S(b_{(3)}) = -(-1)^{|b_{(2)}||a_{(3)}|}\{S(a_{(1)}), S(b_{(1)})\}a_{(2)}b_{(2)}S(a_{(3)})S(b_{(3)}) \\ &= -\{S(a_{(1)}), S(b_{(1)})\}a_{(2)}S(a_{(3)})b_{(2)}S(b_{(3)}) = -\{S(a_{(1)}), S(b_{(1)})\}\varepsilon(a_{(2)})\varepsilon(b_{(2)}) \\ &= -\{S(a_{(1)}\varepsilon(a_{(2)})), S(b_{(1)}\varepsilon(b_{(2)}))\} = -\{S(a), S(b)\} = (-1)^{|a||b|}\{S(b), S(a)\}. \end{aligned}$$

Finally, it only remains to show that $dS = Sd$. Note that

$$\begin{aligned} dS(a) &= dS(a_{(1)}\varepsilon(a_{(2)})) = d(S(a_{(1)})\varepsilon(a_{(2)})) \\ &= d(S(a_{(1)})a_{(2)}S(a_{(3)})) = dS(a_{(1)})a_{(2)}S(a_{(3)}) + (-1)^{|a_{(1)}|}S(a_{(1)})d(a_{(2)}S(a_{(3)})) \\ &= dS(a_{(1)})a_{(2)}S(a_{(3)}) + (-1)^{|a_{(1)}|}S(a_{(1)})(d(a_{(2)})S(a_{(3)}) + (-1)^{|a_{(2)}|}a_{(2)}dS(a_{(3)})) \\ &= dS(a) + (-1)^{|a_{(1)}|}S(a_{(1)})d(a_{(2)})S(a_{(3)}) + (-1)^{|a_{(1)}a_{(2)}|}dS(a), \end{aligned}$$

we get

$$dS(a) = -(-1)^{|a_{(2)}|}S(a_{(1)})d(a_{(2)})S(a_{(3)}).$$

Observe that

$$\Delta d(a) = d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|}a_{(1)} \otimes d(a_{(2)}),$$

we can see

$$0 = \varepsilon d(a)1_A = \eta \varepsilon d(a) = d(a)_{(1)}S(d(a)_{(2)}) = d(a_{(1)})S(a_{(2)}) + (-1)^{|a_{(1)}|}a_{(1)}Sd(a_{(2)}),$$

thus

$$d(a_{(1)})S(a_{(2)}) = -(-1)^{|a_{(1)}|}a_{(1)}Sd(a_{(2)}).$$

Therefore,

$$dS(a) = -(-1)^{|a_{(2)}|}S(a_{(1)})d(a_{(2)})S(a_{(3)}) = S(a_{(1)})a_{(2)}Sd(a_{(3)}) = Sd(a),$$

as claimed. \square

Lemma 3.9. *If $(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ is a DG Poisson Hopf algebra, then for all $a \in A$, $\{a_{(1)}, S(a_{(2)})\} = 0$ if and only if $\{S(a_{(1)}), a_{(2)}\} = 0$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$.*

Proof. By the definition of a DG Poisson Hopf algebra, A is graded commutative graded Hopf algebra, then we have $S \cdot S = I$ by Lemma 2.6. Note that

$$\{a_{(1)}, S(a_{(2)})\} = \{S \cdot S(a_{(1)}), S(a_{(2)})\} = (-1)^{|a_{(1)}||a_{(2)}|} S(\{a_{(2)}, S(a_{(1)})\}) = -S(\{S(a_{(1)}), a_{(2)}\})$$

by Lemma 3.8. It is easy to see sufficiency is true. Similarly, we can prove necessity by the same way. Therefore, we are done. \square

Let us consider the bialgebra structure of universal enveloping algebras for DG Poisson bialgebras (note: the degree of Poisson bracket is zero). Let $(A, \cdot, \{\cdot, \cdot\}, d)$ be a DG Poisson algebra. Define a \mathbb{k} -linear map $\{\cdot, \cdot\}_1$ of degree 0 on A by

$$\{a, b\}_1 = (-1)^{|a||b|} \{b, a\} = -\{a, b\},$$

for all $a, b \in A$. Then $A_1 = (A, \cdot, \{\cdot, \cdot\}_1, d)$ is a DG Poisson algebra by Example 3.3. For a DG algebra (B, \cdot, d) , we denote by $B^{op} = (B, \circ, d)$ the DG opposite algebra of B , where $a \circ b = (-1)^{|a||b|} b \cdot a = a \cdot b$.

Proposition 3.10. [8] Let (A^e, m, h) be the universal enveloping algebra for a DG Poisson algebra $(A, \cdot, \{\cdot, \cdot\}, d)$. Then (A^{eop}, m, h) is the universal enveloping algebra for $A_1 = (A, \cdot, \{\cdot, \cdot\}_1, d)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since Δ is a DG Poisson homomorphism and $(A^e \otimes A^e, m \otimes m, m \otimes h + h \otimes m)$ is the universal enveloping algebra of a DG Poisson algebra $A \otimes A$ (see Lemmas 2.2 and 2.4), there exists a unique DG algebra homomorphism $\Delta^e : A^e \rightarrow A^e \otimes A^e$ such that $\Delta^e m = (m \otimes m)\Delta$ and $\Delta^e h = (m \otimes h + h \otimes m)\Delta$ by Lemma 2.5. Similarly, there exists a DG algebra homomorphism ε^e from A^e into \mathbb{k} such that $\varepsilon^e m = \varepsilon$ and $\varepsilon^e h = 0$, since $(\mathbb{k}, I_{\mathbb{k}}, 0)$ is the universal enveloping algebra of the field \mathbb{k} with trivial differential and trivial Poisson bracket.

Now we claim that $(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e, d^e)$ is a DG bialgebra. Note that (A^e, u^e, η^e, d^e) is a DG algebra. Next, we show that $(A^e, \Delta^e, \varepsilon^e, d^e)$ is a DG coalgebra. Since $(A, \Delta, \varepsilon, d)$ is a DG coalgebra, we have $a_{(11)} \otimes a_{(12)} \otimes a_{(2)} = a_{(1)} \otimes a_{(21)} \otimes a_{(22)}$, $\varepsilon(a_{(1)})a_{(2)} = a = a_{(1)}\varepsilon(a_{(2)})$, $\varepsilon d = 0$ and $\Delta d(a) = d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|} a_{(1)} \otimes d(a_{(2)})$, for all $a \in A$. Thus the coassociativity is immediate from

$$\begin{aligned} (\Delta^e \otimes I)\Delta^e m(a) &= (\Delta^e \otimes I)(m \otimes m)\Delta(a) = \Delta^e m(a_{(1)}) \otimes m(a_{(2)}) \\ &= (m \otimes m)\Delta(a_{(1)}) \otimes m(a_{(2)}) = m(a_{(11)}) \otimes m(a_{(12)}) \otimes m(a_{(2)}), \end{aligned}$$

$$\begin{aligned} (I \otimes \Delta^e)\Delta^e m(a) &= (I \otimes \Delta^e)(m \otimes m)\Delta(a) = m(a_{(1)}) \otimes \Delta^e m(a_{(2)}) \\ &= m(a_{(1)}) \otimes (m \otimes m)\Delta(a_{(2)}) = m(a_{(1)}) \otimes m(a_{(21)}) \otimes m(a_{(22)}), \end{aligned}$$

$$\begin{aligned} (\Delta^e \otimes I)\Delta^e h(a) &= (\Delta^e \otimes I)(m \otimes h + h \otimes m)\Delta(a) \\ &= \Delta^e m(a_{(1)}) \otimes h(a_{(2)}) + \Delta^e h(a_{(1)}) \otimes m(a_{(2)}) \\ &= (m \otimes m)\Delta(a_{(1)}) \otimes h(a_{(2)}) + (m \otimes h + h \otimes m)\Delta(a_{(1)}) \otimes m(a_{(2)}) \\ &= m(a_{(11)}) \otimes m(a_{(12)}) \otimes h(a_{(2)}) + m(a_{(11)}) \otimes h(a_{(12)}) \otimes m(a_{(2)}) \\ &\quad + h(a_{(11)}) \otimes m(a_{(12)}) \otimes m(a_{(2)}) \end{aligned}$$

and

$$\begin{aligned}
(I \otimes \Delta^e) \Delta^e h(a) &= (I \otimes \Delta^e)(m \otimes h + h \otimes m) \Delta(a) \\
&= m(a_{(1)}) \otimes \Delta^e h(a_{(2)}) + h(a_{(1)}) \otimes \Delta^e m(a_{(2)}) \\
&= m(a_{(1)}) \otimes (m \otimes h + h \otimes m) \Delta(a_{(2)}) + h(a_{(1)}) \otimes (m \otimes m) \Delta(a_{(2)}) \\
&= m(a_{(1)}) \otimes m(a_{(21)}) \otimes h(a_{(22)}) + m(a_{(1)}) \otimes h(a_{(21)}) \otimes m(a_{(22)}) \\
&\quad + h(a_{(1)}) \otimes m(a_{(21)}) \otimes m(a_{(22)}).
\end{aligned}$$

Similarly, we can prove the counitary property of A is also true. In order to prove d is a coderivation of degree 1, first we have $\varepsilon^e d^e m(a) = \varepsilon^e m d(a) = \varepsilon d(a) = 0$ and $\varepsilon^e d^e h(a) = \varepsilon^e h d(a) = 0$, and then we should prove $\Delta^e d^e = (d^e \otimes I + T(d^e \otimes I)T) \Delta^e$, which follows from

$$\begin{aligned}
(d^e \otimes I + T(d^e \otimes I)T) \Delta^e m(a) &= (d^e \otimes I + T(d^e \otimes I)T)(m \otimes m) \Delta(a) \\
&= d^e m(a_{(1)}) \otimes m(a_{(2)}) + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes d^e m(a_{(2)}) \\
&= m d(a_{(1)}) \otimes m(a_{(2)}) + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes m d(a_{(2)}),
\end{aligned}$$

$$\begin{aligned}
\Delta^e d^e m(a) &= (m \otimes m) \Delta d(a) \\
&= (m \otimes m)(d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|} a_{(1)} \otimes d(a_{(2)})) \\
&= m d(a_{(1)}) \otimes m(a_{(2)}) + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes m d(a_{(2)}),
\end{aligned}$$

$$\begin{aligned}
\Delta^e d^e h(a) &= (m \otimes h + h \otimes m) \Delta d(a) \\
&= (m \otimes h + h \otimes m)(d(a_{(1)}) \otimes a_{(2)} + (-1)^{|a_{(1)}|} a_{(1)} \otimes d(a_{(2)})) \\
&= m d(a_{(1)}) \otimes h(a_{(2)}) + h d(a_{(1)}) \otimes m(a_{(2)}) \\
&\quad + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes h d(a_{(2)}) + (-1)^{|a_{(1)}|} h(a_{(1)}) \otimes m d(a_{(2)})
\end{aligned}$$

and

$$\begin{aligned}
(d^e \otimes I + T(d^e \otimes I)T) \Delta^e h(a) &= (d^e \otimes I + T(d^e \otimes I)T)(m \otimes h + h \otimes m) \Delta(a) \\
&= d^e m(a_{(1)}) \otimes h(a_{(2)}) + d^e h(a_{(1)}) \otimes m(a_{(2)}) \\
&\quad + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes d^e h(a_{(2)}) + (-1)^{|a_{(1)}|} h(a_{(1)}) \otimes d^e m(a_{(2)}) \\
&= m d(a_{(1)}) \otimes h(a_{(2)}) + h d(a_{(1)}) \otimes m(a_{(2)}) \\
&\quad + (-1)^{|a_{(1)}|} m(a_{(1)}) \otimes h d(a_{(2)}) + (-1)^{|a_{(1)}|} h(a_{(1)}) \otimes m d(a_{(2)}).
\end{aligned}$$

Hence $(A^e, d^e, \Delta^e, \varepsilon^e)$ is a DG coalgebra. Note that Δ^e and ε^e are DG algebra homomorphisms, which means $(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e)$ is a graded bialgebra. Therefore, we finish the proof. \square

In general, A^e is just a DG bialgebra. But in some special cases, A^e can be endowed with the Hopf structure, such that $(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e, S^e, d^e)$ becomes a DG Hopf algebra. Thus, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Since the antipode S is a DG Poisson homomorphism from A into A_1 by Lemmas 2.6 and 3.8, there is a DG algebra homomorphism $S^e : A^e \rightarrow A^{eop}$ such that $S^e m = mS$ and $S^e h = hS$ by Lemma 2.5 and Proposition 3.10.

As for part (2), since $(A^e, u^e, \eta^e, \Delta^e, \varepsilon^e, d^e)$ is a DG bialgebra by Theorem 1.1, it suffices to show that S^e is an antipode for A^e if and only if $\{S(a_{(1)}), a_{(2)}\} = 0$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in A$. In fact, we have

$$\eta^e \varepsilon^e m(a) = \eta^e \varepsilon(a) = \varepsilon(a) 1_{A^e}, \quad \eta^e \varepsilon^e h(a) = 0,$$

$$\begin{aligned}
u^e(S^e \otimes I)\Delta^e m(a) &= u^e(S^e \otimes I)(m \otimes m)\Delta(a) \\
&= S^e m(a_{(1)})m(a_{(2)}) = mS(a_{(1)})m(a_{(2)}) \\
&= m(S(a_{(1)})a_{(2)}) = m(\eta\varepsilon(a)) = \varepsilon(a)1_{A^e}
\end{aligned}$$

and

$$\begin{aligned}
u^e(S^e \otimes I)\Delta^e h(a) &= u^e(S^e \otimes I)(m \otimes h + h \otimes m)\Delta(a) = S^e m(a_{(1)})h(a_{(2)}) + S^e h(a_{(1)})m(a_{(2)}) \\
&= mS(a_{(1)})h(a_{(2)}) + hS(a_{(1)})m(a_{(2)}) = h(S(a_{(1)})a_{(2)}) + m(\{S(a_{(1)}), a_{(2)}\}) \\
&= h(\eta\varepsilon(a)) + m(\{S(a_{(1)}), a_{(2)}\}) = \varepsilon(a)h(1_A) + m(\{S(a_{(1)}), a_{(2)}\}) = m(\{S(a_{(1)}), a_{(2)}\}),
\end{aligned}$$

which imply that $u^e(S^e \otimes I)\Delta^e = \eta^e \varepsilon^e$ if and only if $\{S(a_{(1)}), a_{(2)}\} = 0$, since m is injective. Note that for all $a \in A$, we have $\{a_{(1)}, S(a_{(2)})\} = 0$ iff $\{S(a_{(1)}), a_{(2)}\} = 0$ by Lemma 3.9. Similarly, we can prove the formula $u^e(I \otimes S^e)\Delta^e = \eta^e \varepsilon^e$ if and only if $\{a_{(1)}, S(a_{(2)})\} = 0$. Therefore, we finish the proof. \square

It is not hard to show that the following two corollaries are also true. Namely,

Corollary 3.11. Let $(A, u_A, \eta_A, \Delta_A, \varepsilon_A, S_A, \{\cdot, \cdot\}_A, d_A)$ and $(B, u_B, \eta_B, \Delta_B, \varepsilon_B, S_B, \{\cdot, \cdot\}_B, d_B)$ be any two DG Poisson Hopf algebras with Poisson brackets of degree 0. Then

- (1) $(A \otimes B)^e = A^e \otimes B^e$ is a DG bialgebra.
- (2) If $\{S_A(a_{(1)}), a_{(2)}\} = 0$ and $\{S_B(b_{(1)}), b_{(2)}\} = 0$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$, $\Delta(b) = b_{(1)} \otimes b_{(2)}$ for all $a \in A$ and $b \in B$, then $(A \otimes B)^e$ is a DG Hopf algebra.

Corollary 3.12. Let $(A, u, \eta, \Delta, \varepsilon, S, \{\cdot, \cdot\}, d)$ be a DG Poisson Hopf algebra. Then

- (1) $(A^{op})^e = (A^e)^{op}$ is a DG bialgebra.
- (2) $(A^{op})^e$ is a DG Hopf algebra if and only if $\{S(a_{(1)}), a_{(2)}\} = 0$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in A$.

Example 3.13. Retain the notations of Example 3.5. From Theorem 1.1, we can see that the universal enveloping algebra B^e is a DG bialgebra. But

$$\{S(z_1), z_2\} = \{S(z), 1\} + \{S(1), z\} + \{S(-2x), y\} = 2y \neq 0,$$

we can't conclude that B^e is a DG Hopf algebra by Theorem 1.2.

Example 3.14. Let (L, d_L) be a finite dimensional DG Lie algebra over \mathbb{k} with standard graded Lie bracket $[\cdot, \cdot]$ and let $(S(L), u, \eta)$ be the graded symmetric algebra of L , i.e.,

$$S(L) := \frac{T(L)}{(a \otimes b - (-1)^{|a||b|} b \otimes a)},$$

for any $a, b \in L$, where u is a multiplication and η is a unit. Fix a \mathbb{k} -basis x_1, x_2, \dots, x_n of L . Note that $S(L)$ is the graded commutative polynomial ring $k[x_1, x_2, \dots, x_n]$. As in Example 3.6, we know $S(L)$ is a DG Poisson algebra and it also has a DG Hopf structure which is compatible with the Poisson bracket. That is, $S(L)$ is a DG Poisson Hopf algebra with structure

$$\{a, b\}_{S(L)} := [a, b], \quad \Delta(a) = a \otimes 1 + 1 \otimes a, \quad \varepsilon(a) = 0, \quad S(a) = -a,$$

for all $a, b \in L$. Observe that the universal enveloping algebra $(S(L)^e, m, h)$ is the DG algebra generated by $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relation

$$\begin{aligned}
x_i x_j &= (-1)^{|x_i||x_j|} x_j x_i, \\
y_i x_j &= (-1)^{|x_i||x_j|} x_j y_i + \{x_i, x_j\}, \\
y_i y_j &= (-1)^{|x_i||x_j|} y_j y_i + \psi(\{x_i, x_j\}),
\end{aligned}$$

for all $i, j = 1, \dots, n$ and m and h are given by $m(x_i) = x_i$ and $h(x_i) = y_i$, respectively, where $\psi : S(L) \rightarrow S(L)\langle y_i | i = 1, \dots, n \rangle$ is a \mathbb{k} -linear map of degree 0 defined by $\psi(x_i) = y_i$ for all $i = 1, \dots, n$. By Lemma 3.7 and Theorem 1.2, the universal enveloping algebra $S(L)^e$ is a DG Hopf algebra with Hopf structure

$$\begin{aligned} \Delta^e(x_i) &= x_i \otimes 1 + 1 \otimes x_i & \Delta^e(y_i) &= y_i \otimes 1 + 1 \otimes y_i \\ \varepsilon^e(x_i) &= 0 & \varepsilon^e(y_i) &= 0 \\ S^e(x_i) &= -x_i & S^e(y_i) &= -y_i, \end{aligned}$$

for all $i = 1, \dots, n$.

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GUO: DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA, ZHEJIANG, 321004 P.R. CHINA
E-mail address: 1448199409@qq.com

HU: DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA, ZHEJIANG, 321004 P.R. CHINA
E-mail address: 2015210420@zjnu.edu.cn

LÜ: DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA, ZHEJIANG 321004, P.R. CHINA
E-mail address: jiafenglv@zjnu.edu.cn

WANG: DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA 19122, USA
E-mail address: xingting@temple.edu